

Tight upper and lower bounds to the eigenvalues and critical parameters of a double well

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Abstract

By means of a suitable rational approximation to the logarithmic derivative of the wavefunction we obtain tight upper and lower bounds to the eigenvalues and critical parameters of the quartic double-well potential.

1 Introduction

The anharmonic oscillator with the potential-energy function $V(x) = m^2x^2 + gx^4$ has been widely studied in many different contexts and one can find hundreds of papers on this model that is also discussed in many textbooks. It is therefore almost impossible to try a satisfactory review of the literature. Here we are mainly interested in some results obtained by Turbiner [1, 2] who discussed both the single well ($m^2 > 0$, $g > 0$) as well as the double well ($m^2 < 0$, $g > 0$) cases. In particular, Turbiner considered the most interesting case in which $E(g_{crit}) = 0$ when $m^2 = -1$. More precisely, in the case of the double well the parameter g is chosen so that the energy equals the value of the potential at the top of the barrier located between the two wells. Turbiner [1] obtained

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$g_{crit} = 0.302405$ for the ground state but it is obvious that one can obtain similar critical potential parameters for every quantum-mechanical energy. He also discussed the closely related potential $V(x) = ax^2 + 2x^2$ for positive and negative values of a [2].

The purpose of this paper is to obtain several critical parameters with sufficient accuracy using the Riccati-Padé method (RPM) that proved suitable for the calculation of accurate eigenvalues of one-dimensional and separable Schrödinger equations [3, 4] (see also [5] for a literature review). The RPM is based on a rational approximation to the logarithmic derivative of the wavefunction. Although the logarithmic derivative of the wavefunction is also the basis for the remarkable variational-perturbation theory proposed by Turbiner both approaches are completely different.

In section 2 we briefly discuss the model, section 3 outlines the main ideas of the RPM in order to make this paper sufficiently self-contained, section 4 contains a discussion of the results and section 5 is devoted to further comments and conclusions.

2 The model

A symmetric double well $V(-x) = V(x)$ exhibits two minima of equal depth $V(x_m)$ at $x = \pm x_m$ and a barrier $V(0) > V(x_m)$ at $x = 0$. The number of states with energies in the interval $V(x_m) < E \leq V(0)$ depends on the shape and depth $|V(0) - V(x_m)|$ of the wells. For concreteness, in what follows we consider the particular case $V(x) = -x^2 + gx^4$, $g > 0$, for which $V(x_m) = -1/(4g)$, $x_m = 1/\sqrt{2g}$ and $V(0) = 0$. This potential also vanishes at $x = \pm 1/\sqrt{g}$. The solutions to the Schrödinger equation

$$\psi''(x) + [E - V(x)]\psi(x) = 0, \quad (1)$$

are either even $\psi(-x) = \psi(x)$ or odd $\psi(-x) = -\psi(x)$. An even state exhibits a stationary point at $x = 0$ because $\psi'(0) = 0$. If we assume that $\psi(0) > 0$ then the second derivative at origin $\psi''(0) = -E\psi(0)$ indicates that the point

$x = 0$ can be either a maximum, a minimum or an inflexion point when $E > 0$, $E < 0$ or $E = 0$, respectively. Since $dE/dg = \langle x^4 \rangle > 0$ we conclude that for every eigenvalue there exists a critical value $g = g_{crit}$ such that $E(g) > 0$ when $g > g_{crit}$, $E(g) < 0$ when $g < g_{crit}$ and $E(g_{crit}) = 0$. This analysis applies to every state $\psi_k(x)$, where $k = 0, 1, \dots$, and, consequently, we will have the corresponding critical value g_k given by $E_k(g_k) = 0$, where $E_0 < E_1 < \dots$ for every value of g . However, in the case of an odd state we always have $\psi(0) = 0$ and $\psi''(0) = 0$ because of parity. As indicated above, Turbinder [1, 2] calculated only the critical value for the ground state.

3 The Riccati-Padé method

In this section we outline the application of the RPM [3, 4] to the eigenvalue equation (1) where $V(x)$ is symmetric about the origin: $V(-x) = V(x)$. Without loss of generality we assume that $V(0) = 0$. If $V(x)$ is analytic at the origin we can expand it in a Taylor series

$$V(x) = \sum_{j=1}^{\infty} V_j x^{2j}. \quad (2)$$

The regularized logarithmic derivative of the eigenfunction

$$f(x) = \frac{s}{x} - \frac{\psi'(x)}{\psi(x)}, \quad (3)$$

where $s = 0$ or $s = 1$ for an even or odd state, respectively, satisfies the Riccati equation

$$f'(x) + \frac{2sf(x)}{x} - f(x)^2 + V(x) - E = 0. \quad (4)$$

Since the term $1/x$ in equation (3) removes the pole of $\psi'(x)/\psi(x)$ at the origin in the case of an odd state, we can expand $f(x)$ in a Taylor series

$$f(x) = \sum_{j=0}^{\infty} f_j x^{2j+1}, \quad (5)$$

for any state (either even or odd). The coefficients f_j can be easily calculated by means of the recurrence relation

$$\begin{aligned} f_n &= \frac{1}{2n+2s+1} \left(\sum_{j=0}^{n-1} f_j f_{n-j-1} + E\delta_{n0} - V_n \right), \quad n = 1, 2, \dots, \\ f_0 &= \frac{E}{2s+1}. \end{aligned} \quad (6)$$

The radius of convergence of the Taylor series (5) is determined by the zero of $\psi(x)$ closest to origin. A better approximation is a rational function or Padé approximant [6] that takes into account all the zeroes of the eigenfunction (poles of $f(x)$). However, instead of a standard Padé approximant we choose the rational approximation $x[M/N](x^2)$, where

$$[M/N](z) = \frac{\sum_{j=0}^M a_j z^j}{\sum_{j=0}^N b_j z^j} = \sum_{j=0}^{M+N+1} f_j(E) z^j + O(z^{M+N+2}). \quad (7)$$

Since we can arbitrarily choose $b_0 = 1$ we are left with $M + N + 1$ coefficients of the rational function and the unknown energy E as independent adjustable parameters. Therefore, in order to satisfy equation (7) the approximate energy should be a root of

$$H_D^d(E) = \begin{vmatrix} f_{M-N+1} & f_{M-N+2} & \cdots & f_{M+1} \\ f_{M-N+2} & f_{M-N+3} & \cdots & f_{M+2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{M+1} & f_{M+2} & \cdots & f_{M+N+1} \end{vmatrix} = 0, \quad (8)$$

where $d = M - N = 0, 1, \dots$ and $D = N + 1 = 2, 3, \dots$ is the dimension of the Hankel determinant $H_D^d(E)$. The RPM is based on the fact that sequences of roots $E^{[D,d]}$, $D = 2, 3, \dots$ of the Hankel determinant converge towards the actual eigenvalues of the Schrödinger equation (1).

In the case of the double-well potential $V(x) = -x^2 + gx^4$ the Hankel determinants depend on both E and g . If we set $E = 0$ then we obtain sequences of roots $g^{[D,d]}$ that converge towards the actual critical values of this potential parameter. It was shown that in the case of a quartic potential there are sequences of roots of the Hankel determinants that converge from above or below

towards the eigenvalues depending on the value of d [3,4]. For this reason, when $E = 0$ we expect to obtain upper and lower bounds to the critical values of g .

4 Results

In order to calculate the eigenvalues or their associated critical parameters we obtain the coefficients $f_j(E, g)$ and the Hankel determinants (8) analytically by means of available computer algebra software and then the roots of $H_D^d(E = 0, g)$. The numerical calculation of the roots is straightforward because the coefficients $f_j(E, g)$ and, consequently, the Hankel determinants are polynomial functions of both E and g .

4.1 Critical parameters

Tables 1-12 show the critical parameters $g_k^{[D,0]}$ and $g_k^{[D,1]}$ for $k = 0, 1, \dots, 11$ and $D \leq 30$. We appreciate that $g_k^{[D,0]} > g_k > g_k^{[D,1]}$ which enables us to estimate the critical values of g with unprecedented accuracy because $\left|g_k^{[D,0]} - g_k^{[D,1]}\right| \approx A_k e^{-\alpha_k D}$, $\alpha_k > 0$. Present results confirm the accurate estimation of Turbiner [1]: $g_0 = 0.302405$. Although the rate of convergence of the upper and lower bounds is exponential and almost independent of k the accuracy of the bounds for a given D decreases with k because the sequence that approaches g_k starts at a determinant dimension D_k that increases with k . The reason is that the number of zeros of the wavefunction increases with k and, therefore, the degree of the polynomial in the denominator of the rational approximation should increase consistently to accommodate them.

We have decided to truncate all the results to 40 digits even when some of them have not converged. One can easily estimate the actual accuracy of a particular result by straightforward comparison of two consecutive terms of the sequence.

4.2 Eigenvalues for a deep well

In the case of a deep well any approximation based on the Taylor expansion of the wavefunction about the top of the barrier is expected to be most inefficient [1, 2]. The RPM is not an exception and its accuracy deteriorates noticeably with the well depth. However, its rate of convergence is so great that it is still a useful approach. In order to illustrate this point we choose the potential $V(x) = ax^2 + 2x^4$ already studied by Turbinder [2] who selected $a = -20$ as an illustrative example of a deep well ($x_m = \sqrt{5}$, $V(x_m) = -50$). Tables 13 and 14 show $E_k^{[D,0]}$ and $E_k^{[D,1]}$ for $k = 0, 1$, respectively, and $D \leq 30$. In this case we appreciate that $E_k^{[D,0]} < E_k < E_k^{[D,1]}$ which enables us to estimate the eigenvalue with remarkable accuracy because $|E_k^{[D,0]} - E_k^{[D,1]}| \approx B_k e^{-\beta_k D}$, $\beta_k > 0$. Once again we confirm the accuracy of the results obtained by Turbinder [2]: $E_0 = -43.7793165$ and $E_1 = -43.77931646$.

If we expand the potential about one of its minima, say $x = x_m$, then there are two unknowns to be determined: E and $f_0 = f(x_m)$. In this case we have to resort to a variant of the RPM that is suitable for nonsymmetric potentials [7] that we do not discuss here in detail. One of the features of the RPM is that the number of roots in the neighbourhood of an eigenvalue increases with D [5]. This fact makes the calculation of the optimal sequence of roots more difficult in the present case because the search should be carried out in a two-dimensional space. Table 15 shows that the RPM does not yield neither E_0 nor E_1 but the average $(E_0 + E_1)/2$ as is typical of other approximations. An asterisk to the right of a result indicates that the Newton-Raphson algorithm exhibited oscillatory behaviour and failed to converge for that particular value of D and d .

4.3 Resonances

For the model potential $V(x) = m^2 x^2 + gx^4$ Turbinder [1, 2] considered the cases $m^2 > 0, g > 0$ (single well) and $m^2 < 0, g > 0$ (double well). Another interesting case is $m^2 > 0, g < 0$ that supports complex eigenvalues, or resonances, when

the boundary conditions are those for outgoing waves in both channels ($x \rightarrow -\infty$ and $x \rightarrow \infty$). Since the RPM does not take the boundary conditions explicitly into account the same Hankel quantization condition (8) also provides these complex eigenvalues or resonances [8]. For example, when $m^2 = 1$ and $g = -0.1$ we obtain the accurate results shown in table 16 for the lowest resonance. In this case the RPM does not provide upper and lower bounds but the accuracy is also remarkable.

5 Further comments and conclusions

The RPM has been applied to a wide variety of one-dimensional and separable models along the years [3–5, 7, 8]. In this paper we show that it is suitable for the accurate calculation of the critical parameters of parity-invariant double-well potentials. In the case of the quartic potential it provides tight upper and lower bounds. The calculation of the eigenvalues reveals another feature of the approach that was not noticed earlier. If we apply the algorithm for symmetric potentials we obtain tight upper and lower bounds to the eigenvalues even for the two lowest ones that are almost degenerate in the case of deep wells. If, on the other hand, we expand about one of the minima and apply the algorithm for non-symmetric potentials [7] we only obtain the average of such energies.

In closing we want to stress once more the remarkable accuracy of the results obtained by Turbiner [1, 2] by means of a variational-perturbation approach based on relatively simple analytical functions.

References

- [1] A. V. Turbiner, Anharmonic oscillator and double-well potential: approximating eigenfunctions, *Lett. Math. Phys.* 74 (2005) 169-180.
- [2] A. V. Turbiner, Double well potential: perturbation theory, tunneling, WKB (beyond instantons), *Int. J. Mod. Phys. A* 25 (2010) 647-658.

- [3] F. M. Fernández, Q. Ma, and R. H. Tipping, Tight upper and lower bounds for energy eigenvalues of the Schrödinger equation, *Phys. Rev. A* 39 (1989) 1605-1609.
- [4] F. M. Fernández, Q. Ma, and R. H. Tipping, Eigenvalues of the Schrödinger equation via the Riccati-Padé method, *Phys. Rev. A* 40 (1989) 6149-6153.
- [5] F. M. Fernández, Accurate calculation of eigenvalues and eigenfunctions. I: Symmetric potentials, 2008. arXiv:0807.0655 [math-ph].
- [6] G. A. Baker Jr., *Essentials of Padé Approximants*, Academic Press, New York, San Francisco, London, 1975).
- [7] F. M. Fernández and R. H. Tipping, The Riccati-Padé quantization method for one-dimensional quantum-mechanical models, *Can. J. Phys.* 74 (1996) 697-700.
- [8] F. M. Fernández, Direct Calculation of Accurate Siegert Eigenvalues, *J. Phys. A* 28 (1995) 4043-4051.

Table 1: Upper and lower bounds to the critical parameter g_0

D	$d = 0$	$d = 1$
2	0.3636964837266539687768291423593657530940	0.2916059217599021545472957231792727847579
3	0.3041355199664415618743078042974272705046	0.3021387219255858306637347934008111792746
4	0.3024440034802361046390034536436998252163	0.3023992995246179375766349606056642038148
5	0.3024056418830358335891966561981441268327	0.3024047655333490990402062426326043560772
6	0.3024048839946083646428716441326131478820	0.3024048682769592726748351355482080000580
7	0.3024048703292784294366065525578006908341	0.3024048700650067865802797647536576226344
8	0.3024048700986220241695241682236655194508	0.3024048700943923832247301685376725595064
9	0.3024048700949194356723287031392559699348	0.3024048700948543313895839076211758857242
10	0.3024048700948623100521380286778238786326	0.3024048700948613392649500863709288207287
11	0.3024048700948614566082862233924157760399	0.3024048700948614425100082233465182363622
12	0.3024048700948614441944313418759149633306	0.3024048700948614439942175162248892570706
13	0.3024048700948614440179014941243477196370	0.3024048700948614440151123201520973863135
14	0.3024048700948614440154394233185815060767	0.3024048700948614440154012115553250387643
15	0.3024048700948614440154056590605573924369	0.3024048700948614440154051431943447821851
16	0.3024048700948614440154052028352994883999	0.3024048700948614440154051959611789569142
17	0.3024048700948614440154051967511775059455	0.3024048700948614440154051966606380877527
18	0.3024048700948614440154051966709874069051	0.3024048700948614440154051966698073464149
19	0.3024048700948614440154051966699415817037	0.3024048700948614440154051966699263465505
20	0.3024048700948614440154051966699280719473	0.3024048700948614440154051966699278769474
21	0.3024048700948614440154051966699278989422	0.3024048700948614440154051966699278964660
22	0.3024048700948614440154051966699278967443	0.3024048700948614440154051966699278967131
23	0.3024048700948614440154051966699278967166	0.3024048700948614440154051966699278967162
24	0.3024048700948614440154051966699278967162	0.3024048700948614440154051966699278967162
25	0.3024048700948614440154051966699278967162	

Table 2: Upper and lower bounds to the critical parameter g_2

D	$d = 0$	$d = 1$
4		0.07249401976717864151512732239458411564730
5	0.07880383087757011137683529737824261495507	0.07753054612942325838337430417287006369320
6	0.07777666090366403931216840109390188166990	0.07773099784924783071543382424333385673545
7	0.07773920540992483611759533631182864744840	0.07773777294782089944699396518698855106312
8	0.07773801660196284966767006960062932769463	0.07773797610483737979072405485459871898530
9	0.07773798269725003135521198565186191728902	0.07773798164410960093923147751655311453369
10	0.07773798180949031979011418493561074967290	0.07773798178392349671164663015473467484454
11	0.07773798178781948377703152449820346966539	0.07773798178723362245540683204039908883939
12	0.07773798178732064652096414095882960096355	0.07773798178730786644740643392685396952654
13	0.07773798178730972347206342063332123749390	0.07773798178730945629590258954930069113686
14	0.07773798178730949438039534093379878518226	0.07773798178730948899872407316304388880808
15	0.07773798178730948975299872864487236421486	0.07773798178730948964809530047846122383613
16	0.07773798178730948966257915559793548142474	0.07773798178730948966059313555714627470520
17	0.07773798178730948966086368372390240697374	0.07773798178730948966082705590320204467144
18	0.07773798178730948966083198555066043390927	0.07773798178730948966083132579721610049862
19	0.07773798178730948966083141362302671731686	0.07773798178730948966083140199133518119823
20	0.07773798178730948966083140352433436503540	0.07773798178730948966083140332323498065259
21	0.07773798178730948966083140334949739651516	0.07773798178730948966083140334608237440235
22	0.07773798178730948966083140334652461846449	0.07773798178730948966083140334646757442932
23	0.07773798178730948966083140334647490443053	0.07773798178730948966083140334647396599301
24	0.07773798178730948966083140334647408571461	0.07773798178730948966083140334647407049300
25	0.07773798178730948966083140334647407242196	0.07773798178730948966083140334647407217829
26	0.07773798178730948966083140334647407220897	0.07773798178730948966083140334647407220512
27	0.07773798178730948966083140334647407220560	0.07773798178730948966083140334647407220554
28	0.07773798178730948966083140334647407220555	0.07773798178730948966083140334647407220555
29	0.07773798178730948966083140334647407220555	0.07773798178730948966083140334647407220555

Table 3: Upper and lower bounds to the critical parameter g_4

D	$d = 0$	$d = 1$
7	0.04637676319118771829167960772482706323306	0.04448266941691055535466619198018892670832
8	0.04486554986896052877360988330875349112100	0.04478721704291366918609339348372248917650
9	0.04480270069395284067189801456884715258892	0.04479971395760984967156162050942791454765
10	0.04480027677096987196489514247594614354609	0.04480017295777737728036026969215278345688
11	0.04480019173271378075989120345689529452345	0.04480018839860184546193232659737398313455
12	0.04480018898073987718712336666213537971711	0.04480018888068847861919524987787867883116
13	0.04480018889763290676251510452250331007217	0.04480018889480251171492689131961124002445
14	0.04480018889526922906221996330640653139816	0.04480018889519319933956676058573340904971
15	0.04480018889520544383601354410532360734255	0.0448001888952034930697627026956396644968
16	0.04480018889520380070164273753041430874870	0.04480018889520375265608819979371438996645
17	0.04480018889520376009117480127738216852773	0.04480018889520375895058839125174559994890
18	0.04480018889520375912411377530928727493355	0.0448001888952037590979222828963030071768
19	0.04480018889520375910184581646734771612385	0.04480018889520375910126228938039327226238
20	0.04480018889520375910134847750070616671897	0.04480018889520375910133583118877440345999
21	0.04480018889520375910133767505507731700176	0.04480018889520375910133740784342599125332
22	0.04480018889520375910133744634213765970226	0.04480018889520375910133744082644719944017
23	0.04480018889520375910133744161242203858736	0.04480018889520375910133744150100303657243
24	0.04480018889520375910133744151671873083494	0.04480018889520375910133744151451271412294
25	0.04480018889520375910133744151482093207536	0.04480018889520375910133744151477806241713
26	0.04480018889520375910133744151478399919048	0.04480018889520375910133744151478318049710
27	0.04480018889520375910133744151478329293722	0.04480018889520375910133744151478327755544
28	0.04480018889520375910133744151478327965163	0.04480018889520375910133744151478327936702
29	0.04480018889520375910133744151478327940553	0.04480018889520375910133744151478327940034
30	0.04480018889520375910133744151478327940103	0.04480018889520375910133744151478327940094

Table 4: Upper and lower bounds to the critical parameter g_6

D	$d = 0$	$d = 1$
9	0.03555789093100854947852001233906087375285	0.03092572893772660886926073199814728234093
10	0.03161133560525730782271008444032283771361	0.03145849368669941826818885316031905637790
11	0.03149011870542296376338502841225772774296	0.03148364917358371728205366708630422335728
12	0.03148494329367438817018069366727460457784	0.03148468923680296515244178265796186430402
13	0.03148473822437121499799204350554556202120	0.03148472893577904807755520184965076889074
14	0.03148473066939209541383045474594042161208	0.03148473035061312995068282335883952417254
15	0.03148473040841264012200055796453306251009	0.03148473039807105133151396352293524948923
16	0.03148473039989824041338340929593557562405	0.03148473039957923859546643481029290114944
17	0.03148473039963430349984031102576795682861	0.03148473039962490047024519342698839474870
18	0.03148473039962648972467472714013522248438	0.03148473039962622373951281024412934796390
19	0.03148473039962626784077105845026744982718	0.03148473039962626059387043871581476604155
20	0.03148473039962626177453805053554733260776	0.03148473039962626158375823527601914981292
21	0.03148473039962626161434349129423137002648	0.03148473039962626160947714346266678266061
22	0.03148473039962626161024580222497003617042	0.03148473039962626161012523650616399469066
23	0.03148473039962626161014402047172021513590	0.0314847303996262616101411289775964461238
24	0.03148473039962626161014156015270460639019	0.03148473039962626161014149176851390225706
25	0.03148473039962626161014150216346828657935	0.03148473039962626161014150059223332966340
26	0.03148473039962626161014150082844005299243	0.03148473039962626161014150079311744772934
27	0.03148473039962626161014150079837272301895	0.03148473039962626161014150079759470472809
28	0.03148473039962626161014150079770933562803	0.03148473039962626161014150079769252461137
29	0.03148473039962626161014150079769497890499	0.03148473039962626161014150079769462216118
30	0.03148473039962626161014150079769467379548	0.03148473039962626161014150079769466635289

 Table 5: Upper and lower bounds to the critical parameter g_8

D	$d = 0$	$d = 1$
12	0.02454606377413199119828689007811326833190	0.02421917801850957375955340047328637544178
13	0.02428661490509848495426723668328039541065	0.02427219192148520285307831828202703716164
14	0.02427518920350644058599111753338208200674	0.02427457554899199037037423016257647520912
15	0.02427469912043926734852197785524518494693	0.0242746746130456616814254590866649680638
16	0.02427467940328849131593270483231916087025	0.02427467847982376530194467187325031595840
17	0.02427467865552462797424190681519693334287	0.02427467862251144116586540572557279217836
18	0.02427467862864074047936428296833975349799	0.02427467862751567792228058363500195486763
19	0.02427467862771994664763439199207617680632	0.02427467862768324470768285411390709785292
20	0.02427467862768977342041040780145606689484	0.02427467862768862316320148384898791864478
21	0.02427467862768882396042257443841404567986	0.02427467862768878921703836739073112772072
22	0.02427467862768879517757388028203141313335	0.02427467862768879416334573266611758647620
23	0.02427467862768879433456546695526061746882	0.02427467862768879430587992894599527275900
24	0.02427467862768879431065060211425612673539	0.02427467862768879430986280571665336624900
25	0.02427467862768879430999200803802694651858	0.02427467862768879430997095825833740269428
26	0.02427467862768879430997436578735672832845	0.02427467862768879430997381759182836736611
27	0.02427467862768879430997390525558585568438	0.02427467862768879430997389131843556420278
28	0.02427467862768879430997389352173737221345	0.02427467862768879430997389317532507421902
29	0.02427467862768879430997389322950042565877	0.02427467862768879430997389322107162171259
30	0.02427467862768879430997389322237643876678	0.02427467862768879430997389322217543063241

Table 6: Upper and lower bounds to the critical parameter g_{10}

D	$d = 0$	$d = 1$
14	0.02047221602266282751456657741834448703861	0.01963359120980542253748626901810922604369
15	0.01977995619382549576732985165234401774013	0.01974693950909705411949585984494718255830
16	0.01975395366584947117993371023280556305747	0.01975247206868859784344760242869068628597
17	0.01975277987117262166852573265229226959880	0.01975271678710470678106421086781565332383
18	0.01975272954367646835097927623308544993621	0.01975272699679626485875877128889725490927
19	0.01975272749910508026589732684965854046262	0.01975272740119146954343985854466858794848
20	0.01975272742006407830784338141483236540174	0.01975272741646549344162513054957393198253
21	0.01975272741714457999855452711001915380620	0.01975272741701770267830233753828310016295
22	0.01975272741704118117756836119192878870243	0.01975272741703687653629231814017786830575
23	0.01975272741703765875505103202559254656262	0.01975272741703751783247422983584944371101
24	0.01975272741703754301058620864013871701965	0.01975272741703753854808514709255666516573
25	0.01975272741703753933289078549121445588845	0.01975272741703753919590212025527989701806
26	0.01975272741703753921964046956525127252133	0.01975272741703753921555577820644864641499
27	0.01975272741703753921625385870378559062189	0.01975272741703753921613534202517155387932
28	0.01975272741703753921615533452549872575039	0.01975272741703753921615198293899204352528
29	0.01975272741703753921615254141912901863930	0.01975272741703753921615244890377596374998
30	0.01975272741703753921615246414220476261028	0.01975272741703753921615246164614317022386

Table 7: Upper and lower bounds to the critical parameter g_1

D	$d = 0$	$d = 1$
2	0.2086996778999803716648562090815857819745	0.1631480430486902406431149910168791819850
3	0.1718969800128251621850466269627633199548	0.1703607047409994702758081751158201171033
4	0.1706205139560892552710851925000027130523	0.170578583038468282330294547098227728383
5	0.1705851234732303691401645911164046595938	0.1705841331044978946845395959683158081996
6	0.1705842793540777621350223723941786562760	0.1705842582207720111749295014232036857427
7	0.1705842612172584531052118710606141112861	0.1705842607994262597452448410639135511219
8	0.1705842608568300570962041004073058923890	0.1705842608490478764194508397061227525251
9	0.1705842608500903205860724454343000132200	0.1705842608499521929400533330273499936642
10	0.1705842608499703147949847504660105310099	0.1705842608499679587616751011174176672218
11	0.1705842608499682625237635665875689738621	0.1705842608499682236606681996943088012139
12	0.1705842608499682285973929807899222041968	0.1705842608499682279744377896817537199688
13	0.1705842608499682280525621246154707680480	0.1705842608499682280428211446751570550038
14	0.1705842608499682280440291352128456950145	0.1705842608499682280438800921911945957748
15	0.1705842608499682280438983932598809251813	0.1705842608499682280438961562092488713937
16	0.1705842608499682280438964284902789575884	0.1705842608499682280438963954837566117108
17	0.1705842608499682280438963994695384958829	0.1705842608499682280438963989899794778859
18	0.1705842608499682280438963990474787501195	0.1705842608499682280438963990406073441846
19	0.1705842608499682280438963990414259180386	0.1705842608499682280438963990413286969484
20	0.1705842608499682280438963990413402105333	0.1705842608499682280438963990413388507761
21	0.1705842608499682280438963990413390109393	0.1705842608499682280438963990413389921218
22	0.1705842608499682280438963990413389943273	0.1705842608499682280438963990413389940694
23	0.1705842608499682280438963990413389940995	0.1705842608499682280438963990413389940960
24	0.1705842608499682280438963990413389940964	0.1705842608499682280438963990413389940963
25	0.1705842608499682280438963990413389940963	0.1705842608499682280438963990413389940963
26	0.1705842608499682280438963990413389940963	0.1705842608499682280438963990413389940963

Table 8: Upper and lower bounds to the critical parameter g_3

D	$d = 0$	$d = 1$
5	0.06658423997291568499134684206889912226812	0.06517643978579263140844752299484901358385
6	0.06545525969547331631663568171916144298109	0.06540187408173481594068149565108956282900
7	0.06541178093769876825737277164051845877719	0.06540999426358348159542255715630073839591
8	0.06541030843742374541892381357813324610058	0.06541025444317428244130444366902325905977
9	0.06541026353211840459137158434656862367146	0.06541026203083056984744763345599166019452
10	0.06541026227455184719285856903449364470416	0.06541026223561071297259902248086393032143
11	0.06541026224174188141376880992617857852887	0.06541026224078959072891751291251930008276
12	0.06541026224093564261132742105590729066078	0.06541026224091350499126239375145555001202
13	0.06541026224091682376029353614954016102345	0.06541026224091633132610343955357326924266
14	0.06541026224091640368935822734476921154305	0.06541026224091639315192254921567203670695
15	0.06541026224091639467325676065672782241449	0.06541026224091639445538878773831124637217
16	0.06541026224091639448635085915507961920407	0.06541026224091639448198260617462957942654
17	0.06541026224091639448259465520064910032909	0.06541026224091639448250946052769307853630
18	0.06541026224091639448252124533611449529231	0.06541026224091639448251962486031173960277
19	0.06541026224091639448251984641940638968848	0.06541026224091639448251981629125127661268
20	0.06541026224091639448251982036686428666309	0.06541026224091639448251981981827720688257
21	0.06541026224091639448251981989176565853991	0.06541026224091639448251981988196632469101
22	0.06541026224091639448251981988326726046886	0.06541026224091639448251981988309528353080
23	0.06541026224091639448251981988311792520262	0.06541026224091639448251981988311495602909
24	0.06541026224091639448251981988311534392275	0.06541026224091639448251981988311529343365
25	0.06541026224091639448251981988311529998220	0.06541026224091639448251981988311529913574
26	0.06541026224091639448251981988311529924479	0.06541026224091639448251981988311529923079
27	0.06541026224091639448251981988311529923258	0.06541026224091639448251981988311529923235
28	0.06541026224091639448251981988311529923238	0.06541026224091639448251981988311529923238
29	0.06541026224091639448251981988311529923238	0.06541026224091639448251981988311529923238

Table 9: Upper and lower bounds to the critical parameter g_5

D	$d = 0$	$d = 1$
7	0.04235342681449874828379645588296849560493	0.04008468238431420706503999644754864395157
8	0.04053492929116994535350218365600472273380	0.04044086981225583569490363708593431083698
9	0.04045975479435392023038061859624314153650	0.04045604655295448418017316688034199143398
10	0.04045675821089881525694159182373550765977	0.04045662444278018054848586537819408671823
11	0.04045664910617543504412152668595151889447	0.04045664463966676394297232250175238597000
12	0.04045664543514700963483422664255168886621	0.04045664529566647699538229268938359251518
13	0.04045664531976836367316867234905644507400	0.04045664531566035784685046592826441898386
14	0.04045664531635156317423426766344794710382	0.04045664531623666821495891297369542396561
15	0.04045664531625554841934268920858151245483	0.04045664531625247944390720501249170050884
16	0.04045664531625297319525699655560538570218	0.04045664531625289453074904143020885598121
17	0.04045664531625290694769780460302975452153	0.04045664531625290500497641557666478625810
18	0.04045664531625290530637799647025575437974	0.04045664531625290525999200685692030298506
19	0.04045664531625290526707618855393056265051	0.04045664531625290526600220208649305693245
20	0.04045664531625290526616387945812489665168	0.04045664531625290526613970441442199842509
21	0.04045664531625290526614329587540741524176	0.04045664531625290526614276563623192474136
22	0.04045664531625290526614284345304123226822	0.04045664531625290526614283209838581067494
23	0.04045664531625290526614283374603863631471	0.04045664531625290526614283350822558832505
24	0.04045664531625290526614283354237364763735	0.04045664531625290526614283353749460735177
25	0.04045664531625290526614283353818837539638	0.04045664531625290526614283353809018357822
26	0.04045664531625290526614283353810401871459	0.04045664531625290526614283353810207782254
27	0.04045664531625290526614283353810234895946	0.04045664531625290526614283353810231123684
28	0.04045664531625290526614283353810231646435	0.04045664531625290526614283353810231574271
29	0.04045664531625290526614283353810231584196	0.04045664531625290526614283353810231582836
30	0.04045664531625290526614283353810231583022	0.04045664531625290526614283353810231582996

Table 10: Upper and lower bounds to the critical parameter g_7

D	$d = 0$	$d = 1$
10	0.02944043545506540844767350198035741685093	0.02925382514375896464671235971372483554623
11	0.02929263011019324957778320977958604205969	0.02928459515933800353350736998850181395998
12	0.02928622095818320661586489613547243254370	0.02928589785226081725795933860791764382483
13	0.02928596094712212155460236641440229752371	0.02928594882668895559114675244822546899524
14	0.02928595111921799816826053773236604219610	0.02928595069189246851138552618063367657143
15	0.02928595077045074072266816097994054501954	0.02928595075619700683165712945846651860227
16	0.02928595075875122972135286911608051280956	0.02928595075829890538280596210753436363205
17	0.02928595075837810986295798740066779417270	0.0292859507583643880075684513209985568635
18	0.02928595075836674155819019733965787007034	0.02928595075836634205850095626467616762134
19	0.02928595075836640926179834040441440539466	0.02928595075836639805781956766424063211803
20	0.02928595075836639990974309652901026078448	0.02928595075836639960614840049473166735312
21	0.02928595075836639965552572641176616526079	0.02928595075836639964755575998040863415492
22	0.02928595075836639964883280531828487269726	0.02928595075836639964862961870287410535980
23	0.02928595075836639964866172824801636371768	0.02928595075836639964865668711007929634103
24	0.02928595075836639964865747357201094142001	0.02928595075836639964865735162377861613877
25	0.02928595075836639964865737042175116707099	0.02928595075836639964865736754058295358149
26	0.02928595075836639964865736797974866929830	0.02928595075836639964865736791316517905794
27	0.02928595075836639964865736792320799178975	0.02928595075836639964865736792170081663888
28	0.02928595075836639964865736792192590717946	0.02928595075836639964865736792189244929585
29	0.02928595075836639964865736792189739977306	0.02928595075836639964865736792189667055062
30	0.02928595075836639964865736792189677750333	0.02928595075836639964865736792189676188285

Table 11: Upper and lower bounds to the critical parameter g_9

D	$d = 0$	$d = 1$
12	0.02329060717345882812929809993469954354839	0.02288168656122881606834380073412630921766
13	0.02296466585093838369363438315345906760748	0.02294670804722385869383190696061532989902
14	0.02295046689463449122205871968561471535248	0.02294969054566492819024936034224687544502
15	0.02294984828211896691375200728655512450953	0.02294981670571586685777002151656828297988
16	0.02294982293735504571626453992935772109552	0.02294982172407890215326351386560285988779
17	0.02294982195726747110669443863071991252322	0.02294982191299831900377208175041359575596
18	0.02294982192130408334069325371494219260989	0.02294982191976321787820458223173065952988
19	0.02294982192004600800576202711717634586091	0.02294982191999464238368090308647173457388
20	0.02294982192000388025314759124176339867059	0.02294982192000223462001097458990789510519
21	0.02294982192000252510040859515776395449364	0.0229498219200024727575548992663346212595
22	0.02294982192000248309331156982426267954702	0.02294982192000248157600760241986117382833
23	0.02294982192000248183505209869665923986167	0.02294982192000248179116130013135788646917
24	0.02294982192000248179854345959064375280489	0.02294982192000248179731060544765190386249
25	0.02294982192000248179751508988595278496814	0.02294982192000248179748139795654468872600
26	0.02294982192000248179748691365471395107790	0.02294982192000248179748601628477222654560
27	0.02294982192000248179748616140306635176449	0.02294982192000248179748613807215578726825
28	0.02294982192000248179748614180187080893006	0.02294982192000248179748614120890980039698
29	0.02294982192000248179748614130267722344290	0.02294982192000248179748614128792630866716
30	0.02294982192000248179748614129023512798250	0.02294982192000248179748614128987552346398

Table 12: Upper and lower bounds to the critical parameter g_{11}

D	$d = 0$	$d = 1$
14	0.01996781952733978066990651885169494601712	0.01872252524814441498777702067663915812970
15	0.01890198638848331888258042805590161397986	0.01886069386249469343558391653645193592948
16	0.01886948779463646969880275156276806629728	0.01886761704220707966640143390518547691489
17	0.01886800824507166015980470286736442365237	0.01886792750167288410437554271348871817449
18	0.01886794394845459026863114135387378164300	0.01886794064000340464095534517769369533121
19	0.01886794129758633264365840899606626347236	0.01886794116838374607914916662761523751379
20	0.01886794119349012426038980610674152516619	0.01886794118866314741294671844248716274723
21	0.01886794118958173342426775990842042714033	0.01886794118940863900110547005121626575133
22	0.01886794118944094780343142406309400748702	0.01886794118943497221768172831049899983764
23	0.01886794118943606768966856387448227211230	0.01886794118943586856954280873755757272623
24	0.01886794118943590446563326063745363934179	0.01886794118943589804591666167271857500859
25	0.01886794118943589918520561482923202470277	0.01886794118943589898452368838333550424238
26	0.01886794118943589901961832711799440636334	0.01886794118943589901352396387709244724591
27	0.01886794118943589901457511041947361581581	0.01886794118943589901439500217258055563250
28	0.01886794118943589901442566578641554737888	0.01886794118943589901442047763434765401488
29	0.01886794118943589901442135016026667735164	0.01886794118943589901442120428100247967571
30	0.01886794118943589901442122853193989182105	0.01886794118943589901442122452280735553630

Table 13: Eigenvalue E_0 of $V(x) = -20x^2 + 2x^4$

D	$d = 0$	$d = 1$
6	-45.48176505716663623746030853874339294793	-43.31505048121806269799121924974007323963
7	-43.93562409115915060703569782152785395376	-43.73294969603763128167544575228979847158
8	-43.79290846027861844357310305720444485022	-43.77551108808928229026247019926137805250
9	-43.78034879437027161346303655319189662654	-43.77904558645083988975394614271026643160
10	-43.77938563684300145774048474622395679774	-43.77929944424936518397005859939559664346
11	-43.77932070326800389143960331807526523524	-43.77931559176863118190405961154233053389
12	-43.77931679174058833577150015312372370423	-43.77931651628984503218700608762334214196
13	-43.77931657819722294399860448861432496436	-43.77931656455787726577858851166259130381
14	-43.77931656750691564181875020351006669623	-43.77931656688052264492602808929547704601
15	-43.77931656701135087269142632700246175155	-43.77931656698445901112942619108928218250
16	-43.77931656698990345125782055132247261168	-43.77931656698881697367559286304162915467
17	-43.77931656698903083010421439365129825553	-43.77931656698898928351091605403643087867
18	-43.77931656698899725461293104654246458294	-43.77931656698899574344803837551937305507
19	-43.77931656698899602667888879332542581751	-43.77931656698899597417221606078658466804
20	-43.77931656698899598380452284334618538122	-43.77931656698899598205519055854216039383
21	-43.77931656698899598236983028579062162137	-43.77931656698899598231376207477055682696
22	-43.77931656698899598232366434613920385196	-43.77931656698899598232193049349880226816
23	-43.7793165669889959823222315755255010996	-43.77931656698899598232217970971274714616
24	-43.77931656698899598232218857562333915175	-43.77931656698899598232218707135367394310
25	-43.77931656698899598232218732474833030185	-43.77931656698899598232218728236042310832
26	-43.77931656698899598232218728940341828025	-43.77931656698899598232218728824080056198
27	-43.77931656698899598232218728843150872694	-43.77931656698899598232218728840041742956
28	-43.77931656698899598232218728840545621676	-43.77931656698899598232218728840464430755
29	-43.77931656698899598232218728840477440128	-43.77931656698899598232218728840475366921
30	-43.77931656698899598232218728840475695571	-43.77931656698899598232218728840475643739

Table 14: Eigenvalue E_1 of $V(x) = -20x^2 + 2x^4$

D	$d = 0$	$d = 1$
6	-44.28978145133939862149419754620613346698	-43.62795163389018669860165633098424087735
7	-43.82616673070973173577840556883805307921	-43.76576795931880241521357215234857485653
8	-43.78312553803835348336577354919495769796	-43.77828451010011593274177269922005910433
9	-43.77958746079137554333463082137428876952	-43.77924739174091404787515603368964260950
10	-43.77933358307886826118081591082050673091	-43.77931232397705691808359150077738077252
11	-43.77931743547205578754257366879262383436	-43.77931623549969136036517949635666556850
12	-43.77931651095044783505835230224087611246	-43.77931644904306266813723614415863395298
13	-43.77931646268240972489009270135986067432	-43.77931645973337103998145392160130025351
14	-43.77931646035976410196849335890222640688	-43.77931646022893586058420780053698848631
15	-43.77931646025582772494455556092372338615	-43.77931646025038328424957290289598406600
16	-43.77931646025146976194486559104801805466	-43.77931646025125590549398887865761953113
17	-43.77931646025129745209161080295437763140	-43.77931646025128948098876629023213347179
18	-43.77931646025129099215381622204479194084	-43.77931646025129070892293632955493647256
19	-43.77931646025129076142961452624986298875	-43.77931646025129075179730674129523337257
20	-43.77931646025129075354663920814516135221	-43.77931646025129075323199944815342215392
21	-43.77931646025129075328806766500827785286	-43.77931646025129075327816539260914168142
22	-43.77931646025129075327989924542997826452	-43.77931646025129075327959816334489803674
23	-43.77931646025129075327965002919009846775	-43.77931646025129075327964116327858382287
24	-43.77931646025129075327964266754840557475	-43.77931646025129075327964241415372284625
25	-43.77931646025129075327964245654163445092	-43.77931646025129075327964244949863854605
26	-43.77931646025129075327964245066125638531	-43.77931646025129075327964245047054820050
27	-43.77931646025129075327964245050163950112	-43.77931646025129075327964245049660071339
28	-43.77931646025129075327964245049741262269	-43.77931646025129075327964245049728252895
29	-43.77931646025129075327964245049730326102	-43.77931646025129075327964245049729997452
30	-43.77931646025129075327964245049730049283	-43.77931646025129075327964245049730041150

Table 15: Eigenvalue of the double well $V(x) = -20x^2 + 2x^4$ from expansion about one of the minima

D	E	f_0
$d = 0$		
3	-43.77931669031571738751481356566038039124	0.2273917394440904379752945788994227972398
4	-43.77931651361968401280584765608342823829	0.2273916971399277265908076160133509098932
5	-43.77931651362014038166816249872705138561	0.2273916971400760682068043142501069051802
6	-43.77931651362014024495550010192866031029	0.2273916971400760353994260712747814338263
7	-43.77931651362014498836136885879631004559	0.2273916971400775105333097837863779501094
8	-43.77931651362014510834900235709469809467	0.2273916971400775478952431605722187981473
9	-43.77931651362014514928353456063329997293	0.2273916971400775606506174073218214534183
10	-43.77931651362014009917520045377859851074	0.2273916971400759871019196828844221848766
11	-43.7793165136201449879032522778299494589	0.2273916971400775103665439032195059408009
$d = 1$		
3	-43.77931651361612431420607712081719445236	0.2273916971387571156369644228290555283616
4	-43.77931651362012751710629596872829894209	0.2273916971400720236355367680185887323197
5	-43.77931651362014416185777976622515513944	0.2273916971400772528329334561526339961353
6	-43.77931651362014167227391733804968489512	0.2273916971400764788942388256767738879596*
7	-43.77931651362014410032754942613776847221	0.2273916971400772338084116146065912034121
8	-43.77931651362014339248155845283235823475	0.2273916971400770132576683468221320558258
9	-43.77931651362014447117488104583915749563	0.2273916971400773493605499071994360392614
10	-43.77931651362014400180312517086275181785	0.2273916971400772031103818128168790570476

Table 16: Lowest resonance for $V(x) = x^2 - 0.1x^4$

D	$\Re E$	$\Im E$
$d = 0$		
3	0.9009045987434388961113826558408929050585	0.006531275248132131969937339072308577550367
4	0.9006785761554979170185849838170952561575	0.006697172917429490409114415779897272309962
5	0.9006728969139677462512941148232908174675	0.006693419931424615406839099221196299082930
6	0.9006729020396943939429810890309258597079	0.006693282383360464347154292684069904696091
7	0.9006729040491209448507241419711280157202	0.006693280869143828507690412010788048372771
8	0.9006729040915603251405575905226281082941	0.006693280875265106991770610308683065847521
9	0.9006729040920144630597265124954551335021	0.006693280875789241683725707577273885929508
10	0.9006729040920151108192341764268998578420	0.006693280875799990838393406602497907148018
11	0.9006729040920150269574289739136914578854	0.006693280875800129218078648202714592941434
12	0.9006729040920150248388637300439191244513	0.006693280875800130271842860245002534919132
13	0.9006729040920150248051243749312897357163	0.006693280875800130269532817035582731535822
14	0.9006729040920150248047249847012756191763	0.006693280875800130269277606514629002483066
15	0.9006729040920150248047216955367124563283	0.006693280875800130269271964600663717648139
16	0.9006729040920150248047216887862151349051	0.006693280875800130269271876206213583967297
17	0.9006729040920150248047216891991689220470	0.006693280875800130269271875092705208083531
18	0.9006729040920150248047216892100957991600	0.006693280875800130269271875081399548665397
19	0.9006729040920150248047216892102850542174	0.006693280875800130269271875081318291974165
20	0.9006729040920150248047216892102877256260	0.006693280875800130269271875081318229210992
21	0.9006729040920150248047216892102877579668	0.006693280875800130269271875081318240835207
22	0.9006729040920150248047216892102877583018	0.006693280875800130269271875081318241118075
23	0.9006729040920150248047216892102877583046	0.006693280875800130269271875081318241122880
24	0.9006729040920150248047216892102877583046	0.006693280875800130269271875081318241122949
25	0.9006729040920150248047216892102877583046	0.006693280875800130269271875081318241122950
26	0.9006729040920150248047216892102877583046	0.006693280875800130269271875081318241122950
$d = 1$		
3	0.9006269386934407149024188879619797335942	0.006692165141048608819080712012111923978095
4	0.9006724819314438873902357659829302302388	0.006692380922317620997669150541899710733084
5	0.9006729133557364622189247126467344851372	0.006693264258310679158403839879649976013988
6	0.9006729044182223618433366709225523876453	0.006693280799696702428168425080430007058535
7	0.9006729040968377563374267222635288479840	0.006693280878551503991618828166689870608647
8	0.9006729040920469156664148684004597890937	0.006693280875882034788729653473581427028750
9	0.9006729040920146849762905383362938705186	0.006693280875801427980509583740830419378055
10	0.9006729040920150100162224583175926763758	0.006693280875800143543938942464596749680557
11	0.9006729040920150245224972608124165923811	0.006693280875800130322255056366508536933007
12	0.9006729040920150248008764220038503344438	0.006693280875800130267961175438433627644701
13	0.9006729040920150248046829813016628536234	0.006693280875800130269230805638311673398573
14	0.900672904092015024804721465964997703386	0.006693280875800130269271136460543267193167
15	0.9006729040920150248047216907377831922129	0.006693280875800130269271864791037012412317
16	0.9006729040920150248047216892849347231816	0.006693280875800130269271874964638556109025
17	0.9006729040920150248047216892117960284237	0.006693280875800130269271875080292909149647
18	0.9006729040920150248047216892103110181208	0.006693280875800130269271875081313332926222
19	0.9006729040920150248047216892102880610630	0.006693280875800130269271875081318292154781
20	0.9006729040920150248047216892102877616995	0.006693280875800130269271875081318243107606
21	0.9006729040920150248047216892102877583365	0.006693280875800130269271875081318241161447
22	0.9006729040920150248047216892102877583048	0.006693280875800130269271875081318241123542
23	0.9006729040920150248047216892102877583046	0.006693280875800130269271875081318241122958
24	0.9006729040920150248047216892102877583046	0.006693280875800130269271875081318241122950
25	0.9006729040920150248047216892102877583046	0.006693280875800130269271875081318241122950